جامعة دهوك
كلية التربية الاساس

ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني
"المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني"
16-17 كانون الاول 2020 (المجلد الثاني)

الجامعة العراقية
مركز البحوث والدراسات

SOME RESULTS OF SECOND ORDER DIFFERENTIAL SUBORDINATION FOR MULTIVALENT FUNCTIONS ASSOCIATED WITH INTEGRAL OPERATOR

Zainab H.Mahmood[1], Aseel A.Hassan[2], Dunia K.Mahdi[3]
Department of Mathematics / College of Science / University of Baghdad/ Baghdad/Iraq

Email: zainab_hd@yahoo.com

Department of Mathematics / College of Science / University of Baghdad/ Baghdad/Iraq

Email: aselamer16@gmail.com

Department of physical / College of Science / University of Baghdad/ Baghdad/Iraq

Email: dr.duniaphysics95@yahoo.com

Abstract : In this paper , we investigate essential problems for subclasses related to known functions . Therefore , we get special cases of our major results in a form of theorems and corollaries and proved .

Keyword: analytic function , Hadamard product, Ruscheweyh derivative,univalent function , Differential subordination.

Introduction

Let $\mathcal{A}(\mathfrak{p})\xi$ denoted the class of functions of the form

$$f(\omega,\xi) = \sum_{k=\mathfrak{p}+1}^{\infty} a_k(\xi)\,\omega^k \qquad (\mathfrak{p} \in \mathbb{N}),$$

which are analytic and $\mathfrak{p}-$valent in the open unit disk $\mathbb{U} = \{\omega;\ \omega \in \mathbb{C} : |\omega| < 1\}$.For the functions f and g in $\mathcal{A}(\mathfrak{p})\xi$, we say that f is subordinate to g in U, and write $f \prec g$ if there exists a function $w(\omega)$ in $\mathbb{U}$ such that $|w(\omega)| < 1$ and $w(0) = 0$ with $f(\omega) = g(w(\omega))$ in U. If f is univalent in U, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset$

$g(\mathbb{U})$.

Let $f \in \mathcal{A}(\mathfrak{p})\xi$ and $g \in \mathcal{A}(\mathfrak{p})\xi$ be given by

$$g(\omega,\xi) = \sum_{k=\mathfrak{p}+1}^{\infty} b_k(\xi)\,\omega^k \qquad n \in \mathbb{N}, \omega \in \mathbb{U}, \text{ f or all } \xi \in \mathbb{U},$$

then the Hadamard product (or convolution) $f \prec g$ of f and g is defined (as usual) by

$$(f * g)(\omega,\xi) = \omega^p + \sum_{k=\omega+1}^{\infty} a_k(\xi) b_k(\xi)\omega^k = (g * f)(\omega,\xi)$$

For a function given by

$$f(\omega,\xi) = \sum_{k=\mathfrak{p}+1}^{\infty} a_k(\xi)\,\omega^k \in A(\mathfrak{p})\xi$$

it follows from

$$\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi) = \phi_{\mathfrak{p}}^{(+)}(a,c;\omega) * f(\omega,\xi), \omega \in \mathbb{U}, \xi \in \overline{\mathbb{U}}$$

that $\lambda > -\mathfrak{p}$ and $a, c \in \mathbb{R}\backslash\omega_0^-$. where

$$\phi_{\mathfrak{p}}(a,c;\omega) * \phi_{\mathfrak{p}}^{(+)}(a,c;\omega) = \frac{\omega^{\mathfrak{p}}}{(1-\omega)^{\lambda+\mathfrak{p}}}$$

and

$$\phi_{\mathfrak{p}}(a,c;\omega) = \omega^{\mathfrak{p}} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k}\omega^{\mathfrak{p}+k}$$

$$\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi)=\omega^{\mathfrak{p}}+\sum_{k=1}^{\infty}\frac{(c)_k(\lambda+\mathfrak{p})_k}{(a)_k(1)_k}\omega^{\mathfrak{p}+k}=\omega^{\mathfrak{p}}{}_2F_1(c,\lambda+\mathfrak{p},a;\omega)*f(\omega,\xi) \tag{1}$$

$\omega \in \mathbb{U}$, from (1) we deduce that,

$$\omega\left(\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi)\right)'=(\lambda+\mathfrak{p})\mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi)-\lambda\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi) \tag{2}$$

We also note that ,

$$\mathfrak{l}_{\mathfrak{p}}^{0}(\mathfrak{p}+1,1)f(\omega,\xi)=\mathfrak{p}\int_0^{\omega}\frac{f(t)}{t}dt$$

$$\mathfrak{l}_{\mathfrak{p}}^{1}(\mathfrak{p},1)f(\omega,\xi)=\frac{\omega f'(\omega,\xi)}{\mathfrak{p}}$$

$$\mathfrak{l}_{\mathfrak{p}}^{2}(\mathfrak{p},1)f(\omega,\xi)=\frac{2\omega f'(\omega,\xi)+\omega^2 f''(\omega,\xi)}{\mathfrak{p}(\mathfrak{p}+1)}$$

$\mathfrak{l}_{p}^{n}(a,a)f(\omega,\xi)=\mathcal{D}^{n+\mathfrak{p}-1}f(\omega,\xi)\,, n\in\mathbb{N}, n>-\mathfrak{p}$(Ruscheweyh derivative).

Definition(1): Let Ω be a set in $\mathbb{C}$ and $\mathfrak{q}\in Q_0\cap\mu[0,p]$and $\lambda>-\mathfrak{p}$. The class of admissible functions $\Phi_k[\Omega,\mathfrak{q}]$ consists of those functions $\emptyset:\mathbb{C}^3\times\mathbb{U}\times\overline{\mathbb{U}}\to\mathbb{C}$ that satisfy the admissibility condition:

$$\emptyset(u,v,w;\omega,\xi)\notin\Omega, \tag{3}$$

whenever

$$u=\mathfrak{q}(\xi)\quad,\quad v=\frac{k\xi\mathfrak{q}'(\xi)+\lambda\mathfrak{q}(\xi)}{\lambda+\mathfrak{p}},(\lambda>-\mathfrak{p}),$$

and

$$\mathcal{R}e\left\{\frac{(\lambda+\mathfrak{p})(\lambda+\mathfrak{p}+1)w+(\lambda+\mathfrak{p})(2\lambda+1)v-\lambda(3\lambda+2)u}{\lambda u-(\lambda+\mathfrak{p})v}\right\}$$

$$\geq kRe\left\{\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}+1\right\}, \quad (4)$$

$$\omega\in\mathbb{U}, \zeta\in\partial\mathbb{U}\backslash E(\mathfrak{q}), \xi\in\overline{\mathbb{U}} \text{ and } k\geq\mathfrak{p}.$$

Theorem(1): Let $\emptyset\in\Phi_k[\Omega,\mathfrak{q}]$. If $f\in\mathcal{A}(\mathfrak{p})\xi$ satisfies

$$\{\emptyset(l_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi), l_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi), l_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi); \omega\in\mathbb{U}, \xi\in\overline{\mathbb{U}}\}\subset\Omega, \quad (5)$$

then $l_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi)\prec\mathfrak{q}(\omega)$.

Proof: By using (1) and (2), we get the equivalent relation

$$l_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi)=\frac{\omega(l_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi))'+\lambda l_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi)}{(\lambda+\mathfrak{p})}. \quad (6)$$

Assume that $F(\omega)=l_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi)$.Then

$$l_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi)=\frac{\omega F'(\omega)+\lambda F(\omega)}{(\lambda+\mathfrak{p})}.$$

Therefore,

$$l_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi)=\frac{\omega(l_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi))'+(\lambda+1)l_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi)}{(\lambda+\mathfrak{p}+1)} \quad (7)$$

then we have by (6) ,

$$\left(\mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi)\right)' = \frac{\omega F''(\omega)+(1+\lambda)F'(\omega)}{(\lambda+\mathfrak{p})}. \qquad (8)$$

So,

$$\mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi)$$

$$= \frac{1}{(\lambda+\mathfrak{p}+1)}\left[\omega\left(\frac{\omega F''(\omega)+(1+\lambda)F'(\omega)}{(\lambda+\mathfrak{p})}\right) + (\lambda+1)\left(\frac{\omega F'(\omega)+\lambda F(\omega)}{(\lambda+\mathfrak{p})}\right)\right]$$

$$= \frac{1}{(\lambda+\mathfrak{p}+1)}\left[\frac{\omega^2F''(\omega)+(\lambda+1)\omega F'(\omega)}{(\lambda+\mathfrak{p})} + \frac{(\lambda+1)\omega F'(\omega)+\lambda(\lambda+1)F(\omega)}{(\lambda+\mathfrak{p})}\right]$$

$$= \frac{1}{(\lambda+\mathfrak{p}+1)}\left[\frac{\omega^2F''(\omega)+2(\lambda+1)\omega F'(\omega)+\lambda(\lambda+1)F(\omega)}{(\lambda+\mathfrak{p})}\right]$$

$$= \frac{1}{(\lambda+\mathfrak{p})(\lambda+\mathfrak{p}+1)}\left(\omega^2F''(\omega)+2(\lambda+1)\omega F'(\omega)+\lambda(\lambda+1)F(\omega)\right) \qquad (9)$$

Let $u = r \quad , v = \frac{s+\lambda r}{(\lambda+\mathfrak{p})} , \quad w = \frac{t+2(\lambda+1)s+\lambda(\lambda+1)r}{(\lambda+p)(\lambda+\mathfrak{p}+1)}$.

Assume that

$$\psi(r,s,t;\omega,\xi) = \emptyset(u,v,w;\omega,\xi)$$
$$= \emptyset\left(r,\frac{s+\lambda r}{(\lambda+\mathfrak{p})},\frac{t+2(\lambda+1)s+\lambda(\lambda+1)r}{(\lambda+\mathfrak{p})(\lambda+\mathfrak{p}+1)};\omega,\xi\right). \qquad (10)$$

By using (6) and (7) , we obtain

$$\psi(F(\omega),\omega F'(\omega),\omega^2F''(\omega);\omega,\xi)$$
$$= \emptyset\left(\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi);\omega,\xi\right). \quad (11)$$

Therefore, by making use (5), we get

$$\psi(F(\omega), \omega F'(\omega), \omega^2 F''(\omega); \omega, \xi) \in \Omega. \qquad (11)$$

Also, by using

$$w = \frac{t + 2(\lambda+1)s + \lambda(\lambda+1)r}{(\lambda+p)(\lambda+p+1)},$$

and by simple calculations ,we get

$$\frac{(\lambda+p)(\lambda+p+1)w - (2\lambda+1)(\lambda+p)v + \lambda^2 u}{(\lambda+p)v - \lambda u} = \frac{t}{s} + 1. \qquad (13)$$

and the admissibility condition for $\emptyset \in \Phi_k[\Omega, q]$ is equivalent to the admissibility condition for ψ then , $F(\omega) \prec q(\omega)$. Hence , we get $l_p^{\lambda}(a,c)f(\omega,\xi) \prec q(\omega)$.

If we assume that $\Omega \neq \mathbb{C}$ is a simply connected domain. So, $\Omega = h(\mathbb{U})$, for some conformal mapping h of $\mathbb{U}$ onto Ω. Assume the class is written as $\Phi_k[h, q]$.Therefore, we conclude immediately the following Theorem.

Theorem(2): Let $\emptyset \in \Phi_k[h, q]$. If $f \in \mathcal{A}(p)\xi$ satisfies

$$\emptyset\left(l_p^{\lambda}(a,c)f(\omega,\xi), l_p^{\lambda+1}(a,c)f(\omega,\xi), l_p^{\lambda+2}(a,c)f(\omega,\xi); \omega, \xi\right) \prec\prec h(\omega), \qquad (14)$$

then $$l_p^{\lambda}(a,c)f(\omega,\xi) \prec q(\omega).$$

The next result is the extension of theorem (1) to the behavior of $\mathfrak{q}$ on $\partial\mathbb{U}$ is not known.

Corollary(1): Let $\Omega\subset\mathbb{C}$, $\mathfrak{q}$ be univalent in $\mathbb{U}$ and $\mathfrak{q}(0)=0$. Let $\emptyset \in \Phi_k[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $\mathfrak{q}_\rho(\omega) = \mathfrak{q}(\rho\omega)$. If $f \in \mathcal{A}(\mathfrak{p})\xi$ satisfies

$$\emptyset\left(\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi); \omega, \xi\right) \in \Omega, \qquad (15)$$

then $\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi) \prec \mathfrak{q}(\omega)$.

Theorem (3): Let h and $\mathfrak{q}$ be univalent in U , with $\mathfrak{q}$ (0)=0 and set $\mathfrak{q}_\rho(\omega) = \mathfrak{q}(\rho\omega)$ and $h_\rho(\omega) = h(\rho\omega)$. Let $\emptyset: \mathbb{C}^3\times\mathbb{U}\times\overline{\mathbb{U}}\to\mathbb{C}$ satisfy one of the following conditions:

(1) $\emptyset \in \Phi_k[\Omega, \mathfrak{q}_\rho]$ for some $\rho(0,1)$ or

(2) there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k[h_\rho, \mathfrak{q}_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}(\mathfrak{p})\xi$ satisfies (14) , then

$$\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi) \prec \mathfrak{q}(\omega).$$

Proof: Case (1): By using Theorem (1) , we get $\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi) \prec \mathfrak{q}_\rho$. Since $\mathfrak{q}_\rho(\omega) \prec \mathfrak{q}(\omega)$, then we get the result.

Case (2): Assume that $F(\omega)= \mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi)$ and $F_\rho(\omega) = F(\rho\omega)$. So,

$$\emptyset\left(F_\rho(\omega), \omega F_\rho'(\omega), \omega^2 F_\rho''(\omega); \rho\omega\right) = \emptyset(F(\rho\omega), \rho\omega F'(\rho\omega), \rho^2\omega^2 F''(\rho\omega); \rho\omega) \in h_\rho(\mathbb{U}).$$

By using Theorem (1) with associated

$\emptyset(F(\omega), \omega F'(\omega), \omega^2 F''(\omega); w(\omega))$

$\in \Omega$, where w is any function mapping from

U onto U, with $w(\omega)= \rho\omega$, we obtain $F_\rho(\omega) \prec q_\rho(\omega)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get $\mathfrak{l}_p^{\lambda}(a,c) f(\omega, \xi) \prec \mathfrak{q}(\omega)$.

The next theorem gives the best dominant of the differential subordination (11).

Theorem (4): Let h be univalent in $\mathbb{U}$ and let $\emptyset: \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. Hypothetical differential equation

$$\emptyset\left(\mathfrak{q}(\omega), \frac{\omega \mathfrak{q}'(\omega) + \lambda \mathfrak{q}(\omega)}{(\lambda + \mathfrak{p})}, \frac{\omega^2 \mathfrak{q}''(\omega) + 2(\lambda + 1)\omega \mathfrak{q}'(\omega) + \lambda(\lambda + 1)\mathfrak{q}(\omega)}{(\lambda + \mathfrak{p})(\lambda + \mathfrak{p} + 1)}; \omega, \xi\right) = h(\omega), \quad (16)$$

It's found a solution $\mathfrak{q}$ with $\mathfrak{q}(0)=0$ ششnd satisfy one of the conditions following:

(1) $\mathfrak{q} \in Q_0$ and $\emptyset \in \Phi_k[h, \mathfrak{q}]$.

(2) $\mathfrak{q}$ is univalent in $\mathcal{U}$ and $\emptyset \in \Phi_k[h, \mathfrak{q}_\rho]$ for some $\rho \in (0,1)$.

(3) $\mathfrak{q}$ is univalent in $\mathcal{U}$ and there exists $\rho_0 \in (0,1)$ such that $\emptyset \in \Phi_k[h_\rho, \mathfrak{q}_\rho]$,

for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}(\mathfrak{p})_{\xi}$ satisfies (14) , then $\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi) \prec \mathfrak{q}(\omega)$ and $\mathfrak{q}$ is the best dominant.

Proof: We get that q is a dominant of (14) by using Theorem (2) and Theorem (3). It is also a solution of (14) as q satisfies (16), and thus q will be dominant by all (14) dominants. Therefore, q is the best (14) dominant.

Definition(2) . Let Ω be a set in $\mathbb{C}$ and M>0 . The Class of Permissible Functions $\Phi_k[\Omega, M]$ consists of those functions $\emptyset: \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ such that

$$\emptyset\left(Me^{i\theta}, \frac{(k+\lambda)Me^{i\theta}}{(\lambda+\mathfrak{p})}, \frac{L+[2(1+\lambda)k+\lambda(\lambda+1)]Me^{i\theta}}{(\lambda+\mathfrak{p})(\lambda+\mathfrak{p}+1)}; \omega, \xi\right) \notin \Omega, \quad (17)$$

whenever $\omega \in \mathbb{U}, \xi \in \overline{\mathbb{U}}$,$k \geq 1$.

Corollary(2): Let $\emptyset \in \Phi_k[\Omega, M]$. If $f \in \mathcal{A}(\mathfrak{p})_{\xi}$ satisfies that

$$\emptyset\left(\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi); \omega, \xi\right) \in \Omega \text{, then } \mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi) \prec M\omega.$$

Corollary(3): Let $\emptyset \in \Phi_k[\Omega, M]$. If $f \in \mathcal{A}(\mathfrak{p})_{\xi}$ satisfies that

$$\left|\emptyset\left(\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi); \omega, \xi\right)\right| < M \text{, then } \left|\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi)\right| < M$$

Corollary(4): Let M>0 , and Let $C(\xi)$ be an analytic function in $\overline{\mathbb{U}}$ with $\mathcal{R}e\{\xi C(\xi)\} \geq 0$ for $\xi \in \partial U$. If $f \in \mathcal{A}(\mathfrak{p})_{\xi}$ satisfies

$$|(\lambda+p)(\lambda+p+1)\mathfrak{l}_p^{\lambda+2}(a,c)f(\omega,\xi)-(\lambda+p)\mathfrak{l}_p^{\lambda+1}(a,c)f(\omega,\xi)-$$
$$\lambda^2\mathfrak{l}_p^{\lambda}(a,c)f(\omega,\xi)+C(\xi)|<\lambda M,$$

then $$|\mathfrak{l}_p^{\lambda}(a,c)f(\omega,\xi)|<M.$$

Proof: From Corollary (2) by taking $\emptyset(u,v,w,\omega,\xi)=(\lambda+p)(\lambda+p+1)w-(\lambda+p)v-\lambda^2u+C(\xi)$ and $\Omega=h(\mathbb{U})$, where $h(\boldsymbol{\omega})=\lambda M\omega$. By Using Corollary (2), we have to demonstrate that:

$\emptyset\in\Phi_k[\Omega,M]$, that is ,the admissible condition (17) is satisfied .We get

$$\left|\emptyset\left(Me^{i\theta},\frac{(k+\lambda)Me^{i\theta}}{(\lambda+p)},\frac{L+[2(1+\lambda)k+\lambda(\lambda+1)]Me^{i\theta}}{(\lambda+p)(\lambda+p+1)};\omega,\xi\right)\right|$$

$$=|L+[2(1+\lambda)k+\lambda(\lambda+1)]Me^{i\theta}-(k+\lambda)Me^{i\theta}-\lambda^2Me^{i\theta}+C(\xi)|$$

$$=|L+(1+2\lambda)kMe^{i\theta}+C(\xi)|$$

$$\geq(1+2\lambda)kM+\mathcal{R}e\left\{Le^{-i\theta}+\mathcal{R}e\{C(\xi)e^{-i\theta}\}\right\}$$

$\geq\lambda M$. We therefore get the outcome by Corollary(3),.

Definition(3): Let Ω be a set in $\mathbb{C}$ and $q\in\mu[0,p]$ with $q'(\boldsymbol{\omega})\neq0$. The class of admissible functions $\Phi'_k[\Omega,q]$consists of those functions $\emptyset:\mathbb{C}^3\times\overline{\mathbb{U}}\times\overline{\mathbb{U}}\to\mathbb{C}$ Satisfying the condition of admissibility:

$$\emptyset(u,v,w;\zeta,\xi)\notin\Omega, \qquad (18)$$

whenever

$$u=\mathfrak{q}(\omega), v=\frac{\frac{1}{m}\omega\mathfrak{q}'(\omega)+\lambda\mathfrak{q}(\omega)}{\lambda+\mathfrak{p}}, (\lambda>-\mathfrak{p}),$$

and

$$Re\left\{\frac{(\lambda+\mathfrak{p})(\lambda+\mathfrak{p}+1)w-(2\lambda+1)(\lambda+\mathfrak{p})v+\lambda^2u}{(\lambda+\mathfrak{p})v-\lambda u}\right\}\geq\frac{1}{m}\mathcal{R}e\left\{\frac{\omega\mathfrak{q}''(\omega)}{\mathfrak{q}'(\omega)}+1\right\}, (19)$$

$\omega\in\mathbb{U}, \zeta\in\partial\mathbb{U}\backslash E(q), \xi\in\overline{\mathbb{U}}$ *and* $m\geq p$.

Theorem(5): Let $\emptyset\in\Phi_k'[h,\mathfrak{q}]$. If $f\in\mathcal{A}(\mathfrak{p})\xi$, $I_p^{\lambda}(a,c)f(\omega,\xi)\in Q_0$ and

$$\emptyset\left(I_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi), I_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi), I_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi);\omega,\xi\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega\subset\{\emptyset\left(I_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi), I_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi), I_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi);\omega\in\mathbb{U},\xi\in\overline{\mathbb{U}}\right)\},$$

implies that

$$\mathfrak{q}(\omega)\prec I_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi).$$

Proof: By (11) and

$$\Omega\subset\{\emptyset\left(I_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi), I_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi), I_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi);\omega\in\mathbb{U},\xi\in\overline{\mathbb{U}}\right),$$

we have $\Omega\subset\{\psi(F(\omega),\omega F'(\omega),\omega^2F''(\omega);\omega,\xi);\omega\in\mathbb{U},\xi\in\overline{\mathbb{U}})\}$.

from

$$u=r\quad, v=\frac{s+\lambda r}{(\lambda+\mathfrak{p})},\quad w=\frac{t+2(\lambda+1)s+\lambda(\lambda+1)r}{(\lambda+\mathfrak{p})(\lambda+\mathfrak{p}+1)},$$

we see that the admissibility for $\emptyset \in \Phi'_k[\Omega, \mathrm{q}]$ is equivalent to admissibility condition for ψ. Hence , $\psi \in \Psi'[\Omega, \mathrm{q}]$ and so we have $\mathrm{q}(\omega) \prec \mathfrak{l}_{\mathfrak{p}}^{\lambda}(a, c)f(\omega, \xi)$.

An immediate consequence of Theorem(5) is the following theorem.

Theorem(6): Let $\mathrm{q} \in \mu[0, \mathfrak{p}]$, h be analytic in $\mathbb{U}$ and $\emptyset \in \Phi'_k[h, \mathrm{q}]$. If $f(\boldsymbol{\omega}) \in \mathcal{A}(\mathfrak{p})\xi$,

$$\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a, c)f(\omega, \xi) \in Q_0$$

and

$$\{\emptyset(\mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a, c)f(\omega, \xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a, c)f(\omega, \xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda}(a, c)f(\omega, \xi); \omega, \xi\}$$

is univalent in U, then

$$h(\omega) \prec\prec \emptyset(\mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a, c)f(\omega, \xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a, c)f(\omega, \xi), \mathfrak{l}_{\mathfrak{p}}^{\lambda}(a, c)f(\omega, \xi); \omega, \xi\}, \quad (20)$$

implies that $\mathrm{q}(z) \prec \mathfrak{l}_{\mathfrak{p}}^{\lambda}(a, c)f(\omega, \xi)$.

Theorem(7): Let h be analytic in $\mathbb{U}$ and $\emptyset: \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the differential equation

$$\emptyset\left(\mathrm{q}(\omega), \frac{\omega \mathrm{q}'(\omega) + \lambda \mathrm{q}(\omega)}{(\lambda + \mathfrak{p})}, \frac{\omega^2 \mathrm{q}''(\omega) + 2(\lambda + 1)\omega \mathrm{q}'(\omega) + \lambda(\lambda + 1)\mathrm{q}(\omega)}{(\lambda + \mathfrak{p})(\lambda + \mathfrak{p} + 1)}; \omega, \xi\right) = h(\omega),$$

has a solution $\mathrm{q} \in Q_0$. If $\emptyset \in \Phi'_k[h, \mathrm{q}]$, $f \in \mathcal{A}(\mathfrak{p})\xi$, $\mathfrak{l}_{p}^{\lambda}(a, c)f(\omega, \xi) \in Q_0$ and

$\{\emptyset\big(\mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi);\omega,\xi\}$ is univalent in $\mathbb{U}$, then

$$h(\omega) \prec\prec \emptyset\big(\mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi);\omega,\xi\}, \quad (21)$$

implies that $\mathfrak{q}(\omega) \prec \mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi)$, and q is the best dominant.

Proof: This theorem's evidence is the same as that of the proof theorem(4)

We obtained the following theorem from Theorem (2) and Theorem (6).

Theorem(8): Let h_1 and $\mathfrak{q}_1$ be analytic functions in $\mathbb{U}$, h_2 be a univalent functions in $\mathbb{U}$, $\mathfrak{q}_2 \in Q_0$ with $\mathfrak{q}_1(0) = \mathfrak{q}_2(0)$ =0 and $\emptyset \in \Phi_k[h_2,\mathfrak{q}_2] \cap \Phi'_k[h_1,\mathfrak{q}_1]$. If $f \in \mathcal{A}(\mathfrak{p})\xi$,$\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi) \in \mu[0,\mathfrak{p}] \cap Q_0$ and

$\{\emptyset\big(\mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi);\omega,\xi\}$ is univalent in $\mathbb{U}$, then

$$h_1(\omega) \prec\prec \emptyset\big(\mathfrak{l}_{\mathfrak{p}}^{\lambda+2}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda+1}(a,c)f(\omega,\xi),\mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi);\omega,\xi\} \prec\prec h_2, \quad (22)$$

implies that $\mathfrak{q}_1(\omega) \prec \mathfrak{l}_{\mathfrak{p}}^{\lambda}(a,c)f(\omega,\xi) \prec \mathfrak{q}_2(\omega)$.

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