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The combined Sumudu transform-Adomian decomposition method for solving systems of linear Fredholm-Volterra integral/integro-differential equations

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Abstract

This paper is used to solve the systems of linear Fredholm-Volterra integral/integro-differential equations by means of the combined sumudu transform with Adomian decomposition process. By use a comparison of the numerical results with the exact solutions to demonstrate the high accuracy of the solution results. The results show that the present method is very straightforward and effective.

Keywords: Systems of linear Fredholm-Volterra integral/integro-differential equations; Sumudu Transform; Adomian decomposition method; Padé approximation; Trapezoidal rule; Simpson rule.

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1. Introduction

Systems of Fredholm and Volterra integral/integro-differential equations play a pivotal role in the fields of science, industrial mathematics, control theory of financial mathematics, and engineering [1].

Watugala [2] introduced a new integral transform in 90's, named the sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems.

G. Adomian [3,4] in the beginning of the 1980's proposed a new and fruitful method so-called the Adomian decomposition method (ADM) for solving (algebraic, differential, partial differential, integral, etc.) equations [5-7]. It has been shown that this method yields a rapid convergence of the solutions series.

Advantages of approximate methods over numerical methods are that can solve the difficult problems [8,9]. Solving problems with approximate analytical methods often helps an engineer or scientist to understand a physical problem better and may help improve future procedures and designs used to solve their problems [8,9,10]. Solve high order FIVPs directly without reducing it into first order system [11-14].

In recent years several researchers have been adopted different techniques for solving the systems of Fredholm and Volterra integral/integro-differential equations, such as Adomian decomposition method (ADM) [15,16], He's homotopy perturbation method (HPM) [17-19], homotopy analysis method (HAM) [20,21], variational iteration method (VIM) [22] and many more. Finally, Al-Hayani [23] combined Laplace transform-Adomian decomposition method to solve nth-order integro-differential equations.

The main objective of this work is to use the Combined Sumudu Transform-Adomian Decomposition Method (CST-ADM) to solve the systems of linear Fredholm-Volterra integral/integro-differential equations.

2. Preliminaries of Sumudu Transform

Definition. The Sumudu transform is defined over the set of functions

$A = \{f(t) \mid \exists M, \tau_1, \tau_2 > 0, \text{such that } |f(t)| < Me^{|t|/\tau_j}, \text{if } t \in (-1)^j \times [0, \infty)\}.$

by the following formula:

$$F(r) = S\{f(t)\}(r) = \int_0^\infty f(us)e^{-s}ds, \quad r \in (-\tau_1, \tau_2) \qquad (2.1)$$

provided the integral exists for some r, we shall refer to $f(t)$ as the original function of $F(r)$ and $F(r)$ as the Sumudu transform of the function $f(t)$. We also refer to $f(t)$ as the inverse Sumudu transform of $F(r)$.

Theorem. Let $f(t) \in A$, and let $F_n(r)$ be the Sumudu transform of the *nth* derivative $f^{(n)}(t)$ of $f(t)$, then for $n \geq 1$,

$$F_n(r) = \frac{F(r)}{r^n} - \sum_{k=0}^{n-1} \frac{f^{(n)}(0)}{r^{n-k}}. \qquad (2.2)$$

For further detail and properties of this transform, see [24,25].

The following are some basic properties of the Sumudu transform:

Linearity

$$S\{c_1 f(t) + c_2 g(t)\} = c_1 S\{f(t)\} + c_2 S\{g(t)\}, \qquad (2.3)$$

Convolution

$$S\{(f * g)(t)\} = rS\{f(t)\} \cdot S\{g(t)\} = rF(r) \cdot G(r), \qquad (2.4)$$

Sumudu–Laplace Duality

$$S\{f(t)\} = \frac{1}{r} \mathfrak{L}\left\{f\left(\frac{1}{r}\right)\right\}, \quad \mathfrak{L}\{f(t)\} = \frac{1}{s} S\left\{f\left(\frac{1}{s}\right)\right\}, \qquad (2.5)$$

where $S\{f(t)\}$ and $\mathfrak{L}\{f(t)\}$ denote the Sumudu and Laplace transforms, respectively.

More properties can be found in [24,25].

3. Applications and numerical results

In this section, we apply CST-ADM to obtain approximate-exact solutions for the Systems of linear Fredholm-Volterra integral/integro-differential equations is illustrated in the following two problems. To show the high accuracy of the solution results compared with the exact solutions, we give the numerical results, absolute error **(AE)**, relative error **(RLE)**, residual error **(REE)** and Padé approximation error **(PAE)** as follows

$$\mathbf{AE} = |\text{Exact Solution} - (\text{CST} - \text{ADM})|$$

$$\mathbf{RLA} = \left|\frac{\text{Exact Solution} - (\text{CST} - \text{ADM})}{\text{Exact Solution}}\right|$$

REE: It is clear that he substituted the approximate solution in the given integral equation.

$$\mathbf{PAE} = |\text{Exact Solution} - \text{Padé Approximation}|$$

The computations associated with the problems were performed using Maple 18 package with a precision of 20 dígits.

Problem 1. Firstly, we consider the following system of linear two Fredholm-Volterra integro-differential equations and solve it by using CST-ADM [26]:

$$\begin{cases} f_1'(t) = -\frac{1}{15}t^2 f_2(t) - \frac{\sin 1}{8}t - \sin t + \int_0^1 \frac{1}{8} t f_1(s)ds + \int_0^t \frac{1}{5} s f_2(s)ds\,, \\ f_2'(t) = \frac{26}{27} - \frac{1}{10} t\sin t + \int_0^1 \frac{1}{9} s f_2(s)ds + \int_0^t \frac{1}{10} t f_1(s)ds, \end{cases} \tag{3.1}$$

with the initial conditions $f_1(0) = 1, f_2(0) = 0$ and the exact solutions are $f_{1,Exact}(t) = \cos t$, and $f_{2,Exact}(t) = t$.

Taking Sumudu transform of both sides of the system (3.1) gives

$$\begin{cases} S\{f_1'(t)\}(r) = S\left\{-\frac{1}{15}t^2 f_2(t) - \frac{\sin 1}{8}t - \sin t\right\} + \frac{1}{8}S\left\{t\int_0^1 f_1(s)ds\right\} \\ \qquad + \frac{1}{5}S\left\{\int_0^t s f_2(s)ds\right\}, \\ S\{f_2'(t)\}(r) = S\left\{\frac{26}{27} - \frac{1}{10}t\sin t\right\} + \frac{1}{9}S\left\{\int_0^1 s f_2(s)ds\right\} \\ \qquad + \frac{1}{10}S\left\{t\int_0^t f_1(s)ds\right\}, \end{cases} \tag{3.2}$$

so that

$$\begin{cases} F_1(r) = 1 - \frac{2}{5}r^4 - \frac{\sin 1}{8}r^2 - \frac{r^2}{1+r^2} + \frac{1}{8}rS\left\{t\int_0^1 f_1(s)ds\right\} + \frac{1}{5}rS\left\{\int_0^t s f_2(s)ds\right\}, \\ F_2(r) = \frac{26}{27}r - \frac{r^3}{5(1+r^2)^2} + \frac{1}{9}rS\left\{\int_0^1 s f_2(s)ds\right\} + \frac{1}{10}rS\left\{t\int_0^t f_1(s)ds\right\}, \end{cases} \tag{3.3}$$

where $S\{f_i(t)\}(r) = F_i(r),\ i = 1,2.$

Firstly, we set the series

$$F_i(r) = \sum_{n=0}^{\infty} F_{i,n}(r)\,, \quad i = 1,2. \tag{3.4}$$

Substituting (3.4) into the system (3.3) and using the recursive relation, we get

$$\begin{cases} F_{1,0}(r) = 1 - \frac{2}{5}r^4 - \frac{\sin 1}{8}r^2 - \frac{r^2}{1+r^2}, \\ F_{2,0}(r) = \frac{26}{27}r - \frac{r^3}{5(1+r^2)^2}, \\ F_{1,n+1}(r) = \frac{1}{8}rS\left\{t\int_0^1 f_{1,n}(s)ds\right\} + \frac{1}{5}rS\left\{\int_0^t s f_{2,n}(s)ds\right\}, \\ F_{2,n+1}(r) = \frac{1}{9}rS\left\{\int_0^1 s f_{2,n}(s)ds\right\} + \frac{1}{10}rS\left\{t\int_0^t f_{1,n}(s)ds\right\}. \end{cases} \tag{3.5}$$

Taking the inverse Sumudu transform on both sides of the system (3.5) gives

$$\begin{cases} f_{1,0}(t) = -\dfrac{1}{60}t^4 - \dfrac{\sin 1}{16}t^2 + \cos t, \\ f_{2,0}(t) = \dfrac{26}{27}t + \dfrac{1}{10}t\cos t - \dfrac{1}{10}\sin t, \end{cases}$$

$$\begin{cases} f_{1,n+1}(t) = S^{-1}\left\{\dfrac{1}{8}rS\left\{t\int_0^1 f_{1,n}(s)ds\right\}\right\} + S^{-1}\left\{\dfrac{1}{5}rS\left\{\int_0^t sf_{2,n}(s)ds\right\}\right\}, \\ f_{2,n+1}(t) = S^{-1}\left\{\dfrac{1}{9}rS\left\{\int_0^1 sf_{2,n}(s)ds\right\}\right\} + S^{-1}\left\{\dfrac{1}{10}rS\left\{t\int_0^t f_{1,n}(s)ds\right\}\right\}, \end{cases} \tag{3.6}$$

where $S^{-1}\{F_{i,n+1}(r)\}(t) = f_{i,n+1}(t)$, $i = 1,2$. Then from the system (3.6) the iterations are

$$f_{1,1}(t) = S^{-1}\left\{\frac{1}{8}rS\left\{t\int_0^1 f_{1,0}(s)ds\right\}\right\} + S^{-1}\left\{\frac{1}{5}rS\left\{\int_0^t sf_{2,0}(s)ds\right\}\right\}$$

$$= \left(\frac{4}{25} - \frac{1}{50}t^2\right)\cos t + \frac{1}{10}t\sin t + \frac{47}{768}\sin(1)\,t^2$$

$$-\frac{4}{25} - \frac{1}{4800}t^2 + \frac{13}{810}t^4,$$

$$f_{2,1}(t) = S^{-1}\left\{\frac{1}{9}rS\left\{\int_0^1 sf_{2,0}(s)ds\right\}\right\} + S^{-1}\left\{\frac{1}{10}rS\left\{t\int_0^t f_{1,0}(s)ds\right\}\right\}$$

$$= \left(-\frac{1}{45}t - \frac{1}{2400}t^5\right)\sin 1 + \frac{1}{10}\sin t - \frac{1}{10}t\cos t + \frac{1}{30}\cos(1)\,t$$

$$+\frac{26}{729}t - \frac{1}{21000}t^7,$$

$$f_{1,2}(t) = S^{-1}\left\{\frac{1}{8}rS\left\{t\int_0^1 f_{1,1}(s)ds\right\}\right\} + S^{-1}\left\{\frac{1}{5}rS\left\{\int_0^t sf_{2,1}(s)ds\right\}\right\}$$

$$= \left(\frac{17303}{921600}t^2 - \frac{1}{2700}t^4 - \frac{1}{672000}t^8\right)\sin(1)$$

$$+\left(-\frac{7}{800}t^2 + \frac{1}{1800}t^4\right)\cos(1) + \left(-\frac{4}{25} + \frac{1}{50}t^2\right)\cos t$$

$$-\frac{1}{10}t\sin t + \frac{4}{25} - \frac{20329}{2073600}t^2 + \frac{13}{21870}t^4 - \frac{1}{9450000}t^{10},$$

$$f_{2,2}(t) = S^{-1}\left\{\frac{1}{9}rS\left\{\int_0^1 sf_{2,1}(s)ds\right\}\right\} + S^{-1}\left\{\frac{1}{10}rS\left\{t\int_0^t f_{1,1}(s)ds\right\}\right\}$$
$$= \left(\frac{29111}{1360800}t + \frac{47}{115200}t^5\right)\sin(1) + \left(\frac{7}{100} - \frac{1}{50}t^2\right)\sin t$$
$$-\frac{13}{405}\cos(1)\,t + \left(-\frac{7}{100}t + \frac{1}{500}t^3\right)\cos t$$
$$+\frac{181919}{137781000}t - \frac{2}{375}t^3 - \frac{1}{720000}t^5 + \frac{13}{283500}t^7,$$

and so on, in this manner the rest of the iterations can be obtained. Thus, the approximate solution in a series form is given by

$$\phi_{1,4}(t) = \sum_{i=0}^{3} f_{1,i}(t) = f_{1,0}(t) + f_{1,1}(t) + f_{1,2}(t) + f_{1,3}(t)$$
$$= \left(\frac{556421}{1548288000}t^2 - \frac{1129}{81648000}t^4 - \frac{1}{32256000}t^8\right)\sin(1)$$
$$+\left(-\frac{451}{2500}t + \frac{9}{1250}t^3\right)\sin t + \left(-\frac{101}{576000}t^2 + \frac{1}{48600}t^4\right)\cos(1)$$
$$+\left(\frac{93}{125} + \frac{131}{2500}t^2 - \frac{1}{2500}t^4\right)\cos t + \frac{32}{125} - \frac{1800890809}{8622028800000}t^2$$
$$-\frac{7081}{8266860000}t^4 - \frac{1}{28125}t^6 - \frac{1}{201600000}t^8 - \frac{1}{255150000}t^{10},$$

$$\phi_{2,4}(t) = \sum_{i=0}^{3} f_{2,i}(t) = f_{2,0}(t) + f_{2,1}(t) + f_{2,2}(t) + f_{2,3}(t)$$
$$= \left(\frac{73719067}{2939328000}t + \frac{16103}{138240000}t^5 - \frac{1}{945000}t^7 - \frac{1}{665280000}t^{11}\right)\sin(1)$$
$$+\left(-\frac{1699}{43740}t - \frac{7}{120000}t^5 + \frac{1}{630000}t^7\right)\cos(1)$$
$$+\frac{297571108543}{297606960000}t - \frac{20761}{311040000}t^5 - \frac{1}{15309000}t^7 - \frac{1}{13513500000}t^{13}.$$

Notice that the noise terms that appear between various components vanish. These series have the closed form as $n \to \infty$ gives $f_{1,Exact}(t) = \cos t$, and $f_{2,Exact}(t) = t$, which is the exact solutions of the system (3.1).

In **Table 3.1** we list the errors of **AE, RLE, REE, PAE[4/4]** and ($\|F_i\|_2, i = 1,2$) obtained by the CST-ADM $(\phi_{i,4}(t), i = 1,2)$ on the interval [0,1] comparison with a numerical treatment of fixed point theorem [26]. **Fig. 3.1** represents the exact solutions $(f_{i,Exact}(t), i = 1,2)$ with the Trapezoidal rules (TRAPi, $i = 1,2$) and the Simpson rules (SIMPi, $i = 1,2$) in the interval $0 \le t \le 1$, twenty points have been used in the Trapezoidal and Simpson rules.

Table 3.1. AE, RLE, REE, and PAE for problem 1

t	*i*	AE	RLE	REE	PAE[4/4]	[26]
0.000	1	0.0	0	0.0	0.0	4.1E-07
	2	0.0	--	0.0	0.0	8.2E-07
0.125	1	1.916E-08	1.931E-08	3.009E-07	1.916E-08	--
	2	4.022E-07	3.218E-06	3.094E-06	4.022E-07	--
0.250	1	8.068E-08	8.327E-08	6.638E-07	8.068E-08	1.7E-06
	2	8.048E-07	3.219E-06	3.098E-06	8.048E-07	7.2E-07
0.375	1	1.966E-07	2.113E-07	1.150E-06	1.966E-07	9.2E-05
	2	1.208E-06	3.223E-06	3.119E-06	1.208E-06	1.5E-06
0.500	1	3.873E-07	4.413E-07	1.825E-06	3.869E-07	7.1E-05
	2	1.617E-06	3.235E-06	3.178E-06	1.617E-06	5.3E-06
0.625	1	6.812E-07	8.400E-07	2.752E-06	6.777E-07	2.6E-04
	2	2.038E-06	3.261E-06	3.314E-06	2.039E-06	8.4E-06
0.750	1	1.116E-06	1.525E-06	4.008E-06	1.094E-06	3.8E-03
	2	2.485E-06	3.313E-06	3.587E-06	2.488E-06	9.5E-06
0.875	1	1.740E-06	2.714E-06	5.682E-06	1.642E-06	6.1E-03
	2	2.980E-06	3.406E-06	4.088E-06	2.997E-06	2.9E-05
1.000	1	2.615E-06	4.840E-06	7.894E-06	2.248E-06	5.0E-03
	2	3.563E-06	3.563E-06	4.954E-06	3.627E-06	5.7E-05
$\|F_1\|_2$		9.943E-07	--	--	--	--
$\|F_2\|_2$		1.938E-06	--	--	--	--

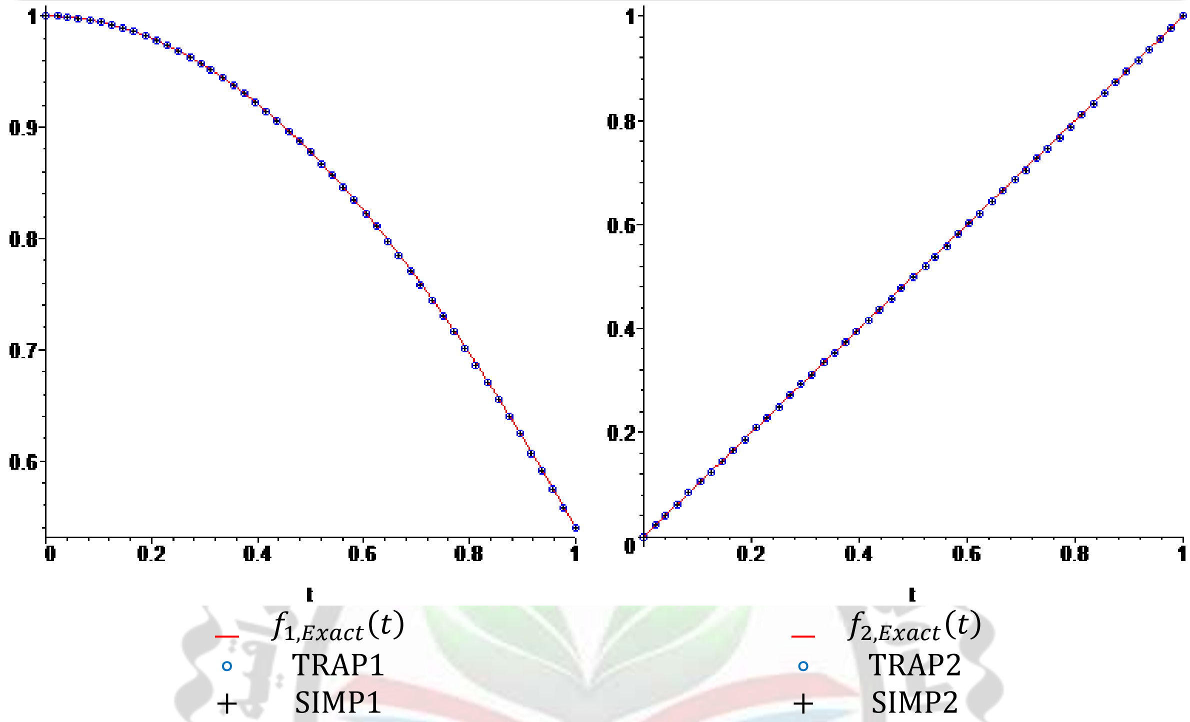

Fig. 3.1. Plots the exact solutions with the Trapezoidal and Simpson rules

Problem 2. Let us consider the following system of linear two Fredholm-Volterra integral equations and solve it by using CST-ADM [27]:

$$\begin{cases} f_1(t) = f_{01}(t) + \int_0^1 \left[\frac{1}{4}(t-x)^3 f_1(s) + \frac{1}{3}(t-x)^2 f_2(s)\right] ds \\ \qquad + \int_0^t \left[\frac{1}{7}(t-x)^4 f_1(s) + \frac{1}{4}(t-x)^3 f_2(s)\right] ds, \\ f_2(t) = f_{02}(t) + \int_0^1 \left[\frac{1}{4}(t-x)^4 f_1(s) + \frac{1}{3}(t-x)^3 f_2(s)\right] ds \\ \qquad + \int_0^t \left[\frac{1}{4}(t-x)^4 f_1(s) + \frac{1}{7}(t-x)^3 f_2(s)\right] ds, \end{cases} \tag{3.7}$$

where $f_{01}(t)$ and $f_{02}(t)$ are chosen such that the exact solutions will be $f_{1,Exact}(t) = t^2$, and $f_{2,Exact}(t) = t^3 + t^2 - t$.

Taking Sumudu transform of both sides of the system (3.7) gives

$$\begin{cases} S\{f_1(t)\}(r) = S\{f_{01}(t)\} + S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^3 f_1(s) + \frac{1}{3}(t-x)^2 f_2(s)\right] ds\right\} \\ \qquad + S\left\{\int_0^t \left[\frac{1}{7}(t-x)^4 f_1(s) + \frac{1}{4}(t-x)^3 f_2(s)\right] ds\right\}, \\ S\{f_2(t)\}(r) = S\{f_{02}(t)\} + S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^4 f_1(s) + \frac{1}{3}(t-x)^3 f_2(s)\right] ds\right\} \\ \qquad + S\left\{\int_0^t \left[\frac{1}{4}(t-x)^4 f_1(s) + \frac{1}{7}(t-x)^3 f_2(s)\right] ds\right\}, \end{cases} \tag{3.8}$$

so that

$$\begin{cases} F_1(r) = F_{01}(r) + S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^3 f_1(s) + \frac{1}{3}(t-x)^2 f_2(s)\right] ds\right\} \\ \qquad + S\left\{\int_0^t \left[\frac{1}{7}(t-x)^4 f_1(s) + \frac{1}{4}(t-x)^3 f_2(s)\right] ds\right\}, \\ F_2(r) = F_{02}(r) + S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^4 f_1(s) + \frac{1}{3}(t-x)^3 f_2(s)\right] ds\right\} \\ \qquad + S\left\{\int_0^t \left[\frac{1}{4}(t-x)^4 f_1(s) + \frac{1}{7}(t-x)^3 f_2(s)\right] ds\right\}, \end{cases} \tag{3.9}$$

where $S\{f_i(t)\}(r) = F_i(r),\ i = 1,2.$

Substituting (3.4) into the system (3.9) and using the recursive relation, we get

$$\begin{cases} F_{1,0}(r) = F_{01}(r), \\ F_{2,0}(r) = F_{02}(r), \\ F_{1,n+1}(r) = S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^3 f_{1,n}(s) + \frac{1}{3}(t-x)^2 f_{2,n}(s)\right] ds\right\} \\ \qquad + S\left\{\int_0^t \left[\frac{1}{7}(t-x)^4 f_{1,n}(s) + \frac{1}{4}(t-x)^3 f_{2,n}(s)\right] ds\right\}, \\ F_{2,n+1}(r) = S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^4 f_{1,n}(s) + \frac{1}{3}(t-x)^3 f_{2,n}(s)\right] ds\right\} \\ \qquad + S\left\{\int_0^t \left[\frac{1}{4}(t-x)^4 f_{1,n}(s) + \frac{1}{7}(t-x)^3 f_{2,n}(s)\right] ds\right\}. \end{cases} \tag{3.10}$$

Taking the inverse Sumudu transform on both sides of the system (3.10) gives

$$\begin{cases} f_{1,0}(t) = f_{01}(t), \\ f_{2,0}(t) = f_{02}(t), \end{cases}$$

$$\begin{cases} f_{1,n+1}(t) = S^{-1}\left\{S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^3 f_{1,n}(s) + \frac{1}{3}(t-x)^2 f_{2,n}(s)\right] ds\right\}\right\} \\ \qquad +S^{-1}\left\{S\left\{\int_0^t \left[\frac{1}{7}(t-x)^4 f_{1,n}(s) + \frac{1}{4}(t-x)^3 f_{2,n}(s)\right] ds\right\}\right\}, \\ f_{2,n+1}(t) = S^{-1}\left\{S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^4 f_{1,n}(s) + \frac{1}{3}(t-x)^3 f_{2,n}(s)\right] ds\right\}\right\} \\ \qquad +S^{-1}\left\{+S\left\{\int_0^t \left[\frac{1}{4}(t-x)^4 f_{1,n}(s) + \frac{1}{7}(t-x)^3 f_{2,n}(s)\right] ds\right\}\right\}, \end{cases} \quad (3.11)$$

where $S^{-1}\{F_{i,n+1}(r)\}(t) = f_{i,n+1}(t),\ i = 1,2$. Then from the system (3.11) the iterations are

$$f_{1,1}(t) = S^{-1}\left\{S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^3 f_{1,0}(s) + \frac{1}{3}(t-x)^2 f_{2,0}(s)\right] ds\right\}\right\}$$
$$+S^{-1}\left\{S\left\{\int_0^t \left[\frac{1}{7}(t-x)^4 f_{1,0}(s) + \frac{1}{4}(t-x)^3 f_{2,0}(s)\right] ds\right\}\right\}$$
$$= \frac{2226967}{27941760} - \frac{455845}{2032128}t + \frac{234079}{1128960}t^2 - \frac{266129}{3386880}t^3 + \frac{13}{2880}t^4$$
$$+\frac{671}{100800}t^5 - \frac{407}{302400}t^6 - \frac{257}{211680}t^7 + \frac{11}{94080}t^8 + \frac{1}{282240}t^9$$
$$-\frac{1}{470400}t^{10} + \frac{1}{1940400}t^{11} + \frac{1}{65197440}t^{12},$$

$$f_{2,1}(t) = S^{-1}\left\{S\left\{\int_0^1 \left[\frac{1}{4}(t-x)^4 f_{1,0}(s) + \frac{1}{3}(t-x)^3 f_{2,0}(s)\right] ds\right\}\right\}$$
$$+S^{-1}\left\{+S\left\{\int_0^t \left[\frac{1}{4}(t-x)^4 f_{1,0}(s) + \frac{1}{7}(t-x)^3 f_{2,0}(s)\right] ds\right\}\right\}$$
$$= \frac{6672773}{93139200} - \frac{9733}{34650}t + \frac{14219}{34560}t^2 - \frac{227903}{846720}t^3$$
$$+\frac{54973}{677376}t^4 + \frac{11}{10080}t^5 + \frac{11}{21600}t^6 + \frac{239}{423360}t^7 + \frac{11}{94080}t^8$$
$$+\frac{1}{493920}t^9 - \frac{19}{6585600}t^{10} + \frac{13}{21732480}t^{11} + \frac{1}{37255680}t^{12},$$

and so on, in this manner the rest of the iterations can be obtained. Thus, the approximate solution in a series form is given by

$$\phi_{1,4}(t)=\sum_{i=0}^{3} f_{1,i}(t)=f_{1,0}(t)+f_{1,1}(t)+f_{1,2}(t)+f_{1,3}(t)$$
$$=0.999998580t^2+(<10^{-5})\cdot t^n, n=0,1,3,4,5,\dots,22,$$

$$\phi_{2,4}(t)=\sum_{i=0}^{3} f_{2,i}(t)=f_{2,0}(t)+f_{2,1}(t)+f_{2,2}(t)+f_{2,3}(t)$$
$$=1.000001375t^3+0.999998227t^2-0.999998464t$$
$$+(<10^{-5})\cdot t^n, n=,4,5,\dots,22.$$

These series have the closed form as $n \to \infty$ gives $f_{1,Exact}(t)=t^2$, and $f_{2,Exact}(t)=t^3+t^2-t$, which is the exact solutions of the system (3.7).

In **Table 3.2** we list the errors of **AE, RLE, REE, PAE[4/4]** and $(\|F_i\|_2, i=1,2)$ obtained by the CST-ADM $\left(\phi_{i,4}(t), i=1,2\right)$ on the interval $[0,1]$ comparison with a numerical treatment of fixed point theorem [27]. **Fig. 3.2** represents the exact solutions $\left(f_{i,Exact}(t), i=1,2\right)$ with the Trapezoidal rules $(\mathrm{TRAP}i, i=1,2)$ and the Simpson rules $(\mathrm{SIMP}i, i=1,2)$ in the interval $0\le t\le 1$, twenty points have been used in the Trapezoidal and Simpson rules.

Table 3.2. AE, RLE, REE, and PAE for problem 2

t	*i*	AE	RLE	REE	PAE[4/4]	[27]
0.0	1	5.758E-07	--	5.719E-07	5.758E-07	2.6E-04
	2	4.953E-07	--	4.935E-07	4.953E-07	3.1E-04
0.1	1	4.551E-07	4.551E-05	4.533E-07	4.551E-07	2.4E-04
	2	3.581E-07	4.024E-06	3.573E-07	3.581E-07	2.8E-04
0.2	1	3.589E-07	8.973E-06	3.579E-07	3.589E-07	2.1E-04
	2	2.486E-07	1.635E-06	2.483E-07	2.486E-07	2.2E-04
0.3	1	2.824E-07	3.138E-06	2.814E-07	2.824E-07	1.7E-04
	2	1.603E-07	8.762E-07	1.604E-07	1.603E-07	1.6E-04
0.4	1	2.209E-07	1.381E-06	2.195E-07	2.209E-07	1.2E-04
	2	8.857E-07	5.032E-07	8.917E-08	8.858E-08	9.8E-05
0.5	1	1.699E-07	6.799E-07	1.680E-07	1.697E-07	7.9E-05
	2	3.094E-07	2.475E-07	3.212E-08	3.104E-08	4.0E-05
0.6	1	1.255E-07	3.488E-07	1.233E-07	1.247E-07	7.1E-05

	2	1.219E-07	5.080E-07	1.049E-08	1.171E-08	3.3E-05
0.7	1	8.478E-08	1.730E-07	8.272E-08	8.225E-08	5.6E-05
	2	3.750E-08	2.819E-07	3.572E-08	3.563E-08	2.0E-05
0.8	1	4.575E-08	7.148E-08	4.458E-08	3.956E-08	3.8E-05
	2	3.838E-08	1.090E-07	3.758E-08	3.240E-08	5.6E-06
0.9	1	7.576E-09	9.353E-09	8.128E-09	5.303E-09	2.1E-05
	2	4.971E-09	7.779E-09	7.070E-09	1.170E-08	1.2E-05
1.0	1	3.017E-08	3.017E-08	2.701E-08	5.338E-08	7.2E-06
	2	7.555E-08	7.555E-08	6.754E-08	1.171E-07	4.1E-05
$\|\|F_1\|\|_2$		2.592E-07	--	--	--	--
$\|\|F_2\|\|_2$		1.841E-07	--	--	--	--

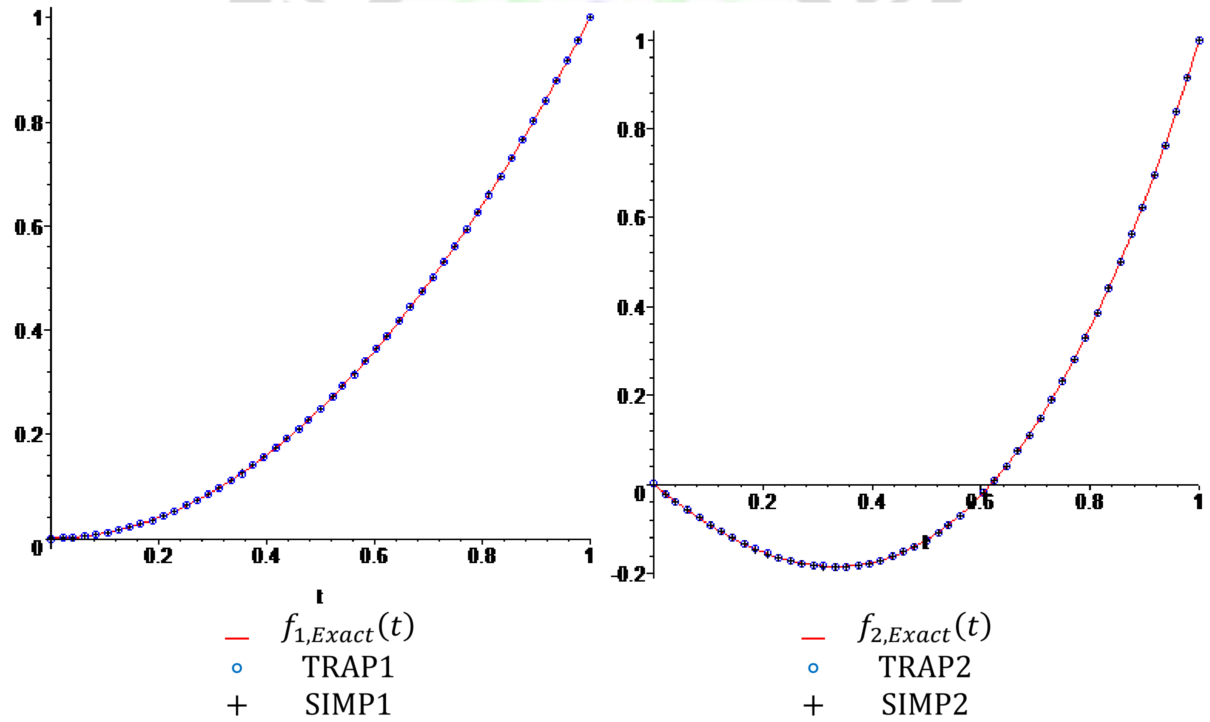

Fig. 3.2. Plots the exact solutions with the Trapezoidal and Simpson rules

Conclusions

In the present paper, the Combined Sumudu Transform-Adomian Decomposition Method has been successfully applied for solving the systems of linear Volterra integral equations. The proposed scheme of CST-ADM has been applied directly without any need for transformation formulae or restrictive assumptions. The solution process of CST-ADM is compatible with those methods in the literature providing analytical approximation such as numerical treatment of fixed-point theorem and numerical methods. The approach of CST-ADM has been tested by employing the method to obtain approximate-exact solutions of two problems. The results obtained in all cases demonstrate the reliability and the efficiency of this method. It has been shown that error is monotically reduced with the increment of the integer n.

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