



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الأول 2020



الجامعة العراقية  
مركز البحوث والدراسات

## Applying the Variational iteration method to initial value problems with discontinuities

Waleed Al-Hayani, Yousif Ali Mohammed

Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Iraq.

E-mail address: waleedalhayani@uomosul.edu.iq / waleedalhayani@yahoo.es

E-mail address: Yousif.csp116@student.uomosul.edu.iq

### Abstract

In this work, the Variational iteration method is practiced to the first order differential equations of initial value problems with Different kinds of discontinuities, giving relevant cases for the two linear and non-linear for different cases.

We make a comparison of the numerical results with the analytic solutions to demonstrate the high accuracy of the solution results. Also, practice the classic Variational iteration method with the iteration of the integral equation and the numerical solution with the Trapezoidal and Simpson rules. The results show that the present method is very straightforward and effective.

**Keywords:** Variational iteration method; Lagrange multiplier ( $\mu$ ); Initial value problems; Unit step function; Unit impulse function; Trapezoidal rule; Simpson rule.

### 1. Introduction

In 1978, Inokuti et al. [1] proposed a general Lagrange multiplier method to solve non-linear problems. In 1999, He [2-4] proposed a new and fruitful method (so called variational iteration method (VIM)) for solving linear and non-linear equations. The VIM has been utilized by many authors quite effectively for obtaining analytic and/or approximate solutions for a wide variety of scientific and engineering applications: linear and non-linear, homogeneous and inhomogeneous as well [5,6]. The VIM gives rapidly convergent successive approximations of the analytic solution if such a solution exists; otherwise, a few approximations can be utilized for numerical purposes. The VIM generates a series solution that occasionally converges to the analytic solution. The idea of the method is based on constructing a correction functional by a general Lagrange multiplier and the multiplier is



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

( المجلد الثاني ) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

chosen in such a way that its correction solution is improved with respect to the initial approximation or to the trial function.

In the modern biography, Waleed and Luis [7-9] practiced the Adomian decomposition method (ADM) to the initial value problems (IVPs) with discontinuities. Al-Hayani and Rasha Fahad [10,11] have been utilized homotopy analysis method (HAM) for the same IVPs. Ji-Huan [12] utilized the homotopy perturbation method (HPM) to solve non-linear oscillators with discontinuities.

The main objective of this paper is to apply the VIM for solving IVPs of first order with a discontinuous derivative, with unit step function or with unit impulse function to obtain approximate-analytic solutions for different cases.

In what follows, we give a brief review of the VIM. We consider the following general non-linear equation

$$Lu(x) + Nu(x) = g(x), \quad (1.1)$$

where  $L$  is the linear operator,  $N$  is the non-linear operator and  $g(x)$  is a known analytic function. The VIM was proposed by He [2-4], where a correction functional for Eq. (1.1) can be rewritten as

$$u_{n+1}(x) = u_n(x) + \int_0^x \mu(s) \{Lu_n(s) + N\tilde{u}_n(s) - g(s)\} ds, n = 0,1,2, \dots \quad (1.2)$$

where  $\mu = \mu(s)$  is general Lagrange multiplier [1-4] which can be identified optimally via variational theory,  $u_0(x)$  is an initial approximation, with possible unknowns. The function  $\tilde{u}_n$  is considered as a restricted variation [13], which means  $\delta\tilde{u}_n = 0$ . Therefore, we first determine the Lagrange multiplier  $\mu$  that will be identified optimally via integration by parts. The successive approximation  $u_{n+1}(x)$ , to the solution  $u(x)$  will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function  $u_0(x)$ , which can be easily solved by mathematical symbolic programs like Maple and Mathematica. Consequently, the approximate-analytic solutions may be obtained by using  $u(x) = \lim_{n \rightarrow \infty} u_n(x)$ .

## 2. VIM applied to an IVP



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

( المجلد الثاني ) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

Consider the general IVP of first order [8,11]:

$$u' + k^2u - N(u) = \lambda f(x, u), \quad u(0) = \alpha, \quad 0 \leq x \leq T, \quad (2.1)$$

where  $k$ ,  $\lambda$  and  $\alpha$  are real constants,  $N$  is a (possibly) non-linear function of  $y$  and  $f$  is a function with some discontinuity.

A correction functional of equation (2.1) is the variational iteration method described iteratively by

$$u_{n+1}(x) = u_n(x) + \int_0^x \mu(s) \{u'_n(s) + k^2u_n(s) - N(\tilde{u}_n(s)) - \lambda f(s, u(s))\} ds, \quad (2.2)$$

where  $\mu = \mu(s)$  is the Lagrange multiplier. Then, we have

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \mu(s) \{u'(s) + k^2u(s) - N(u(s)) - \lambda f(s, u(s))\} ds, \quad (2.3)$$

Calculus of variations and integration by parts, and noting that  $\delta \tilde{u}_n(0) = 0$ , we obtain the following stationary conditions

$$\begin{cases} 1 + \mu(s)|_{s=x} = 0, \\ \mu'(s)|_{s=x} = 0, \end{cases}$$

Solving this system of equations for  $\mu(s, x)$  yields  $\mu(s) = -1$ , and the following iteration formula can be obtained

$$u_{n+1}(x) = u_n(x) - \int_0^x \{u'(s) + k^2u(s) - N(u(s)) - \lambda f(s, u(s))\} ds, \quad (2.4)$$

In what follows, we will apply the VIM for linear and non-linear cases.

**2.1 Linear case:** Let  $g(u) = 0$  and  $\alpha = 1$ .

**2.1.1.** If we are taking  $\lambda = 10$ ,  $k = 2$  and the function  $f(x, u)$  is continuously, but not differentiable, for instance

$$f(x, u) = \begin{cases} -x + \frac{1}{2}, & \text{if } x < \frac{1}{2} \\ x - \frac{1}{2}, & \text{if } x \geq \frac{1}{2} \end{cases}$$

We calculate the first iterations from the Eq. (2.4)

$$u_0 = 1,$$



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

"المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني"

(المجلد الثاني) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

$$u_1(x) = \begin{cases} 1 + x - 5x^2, & x < \frac{1}{2} \\ \frac{7}{2} - 9x + 5x^2, & x \geq \frac{1}{2} \end{cases}$$

$$u_2(x) = \begin{cases} 1 + x - 7x^2 + \frac{20}{3}x^3, & x < \frac{1}{2} \\ \frac{31}{6} - 19x + 23x^2 - \frac{20}{3}x^3, & x \geq \frac{1}{2} \end{cases}$$

$$u_3(x) = \begin{cases} 1 + x - 7x^2 + \frac{28}{3}x^3 - \frac{20}{3}x^4, & x < \frac{1}{2} \\ 6 - \frac{77}{3}x + 43x^2 - \frac{92}{3}x^3 + \frac{20}{3}x^4, & x \geq \frac{1}{2} \end{cases}$$

⋮

thus, in this way, the rest of the iterations can be obtained. When  $n \rightarrow \infty$ , the closed form is

$$u_{Exact}(x) = \begin{cases} \frac{15}{8} - \frac{5}{2}x - \frac{7}{8}e^{-4x}, & x < \frac{1}{2} \\ -\frac{15}{8} + \frac{5}{2}x - \frac{7}{8}e^{-4x} + \frac{5}{4}e^{-4x+2}, & x \geq \frac{1}{2} \end{cases} \quad (2.5)$$

which is the analytic solution for the case **2.1.1**.

In **Table 1** show the comparison of the numerical results applying the VIM ( $n = 15$ ), the numerical solution of (2.4) with the Trapezoidal rule (**TRAP**) (2.4) and the numerical solution of (2.4) with the Simpson rule (**SIMP**) with the analytic solution (2.5) and HAM [11]. Twenty points have been utilized in the Trapezoidal rule and the Simpson rule. It can be noticed that the results obtained by the present method are compatible with the HAM [11]. **Table 2** we have included the maximum absolute error (**MAE**),  $\|\cdot\|_2$ , the maximum relative error (**MRE**) and the maximum residual error (**MRR**) obtained by the **VIM** with the analytic solution (2.5) on the interval  $[0,1]$ . The estimated orders of convergence (EOC) for different values of the constant  $k$  are given in **Table 3**. **Fig. 1** illustrates both the analytic solution  $u_{Exact}(x)$  and our approximation by VIM ( $n = 13$ ) for  $\lambda = 100$  and  $k = 2$  within the interval  $0 \leq x \leq 1$ . For  $k \geq 3$ , the VIM implementation requires approaches to the  $n \geq 20$  if we want to reach beyond the discontinuity (at  $x = \frac{1}{2}$ ).





جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

( المجلد الثاني ) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

Table 1. Numerical results for the case 2.1.1

$x$	$u_{Exact}(x)$	VIM	HAM [11]	TRAP	SIMP
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.038469959	1.038469959	1.038469959	1.038469959	1.038469959
0.2	0.981837156	0.981837156	0.981837156	0.981837156	0.981837156
0.3	0.861455064	0.861455064	0.861455064	0.861455064	0.861455064
0.4	0.698340546	0.698340546	0.698340546	0.698340546	0.698340546
0.5	0.506581627	0.506581627	0.506581623	0.506581627	0.506581627
0.6	0.383521848	0.383521856	0.383521796	0.383521856	0.383521856
0.7	0.383452400	0.383452485	0.383451937	0.383452481	0.383452481
0.8	0.465825836	0.465826473	0.465822862	0.465826434	0.465826436
0.9	0.603462390	0.603466035	0.603447544	0.603465771	0.603465783
1.0	0.778142920	0.778159723	0.778082740	0.778158317	0.778158381

Table 2. MAE,  $\| \cdot \|_2$ , MRE and MRR for the case 2.1.1

$m$	MAE	$\  \cdot \ _2$	MRE	MRR
8	4.117E-03	1.705E-03	7.565E-03	1.735E-00
9	3.849E-03	1.410E-03	5.442E-03	2.813E-02
10	2.940E-03	7.841E-04	3.778E-03	2.652E-02
11	1.448E-03	3.366E-04	1.861E-03	1.755E-02
12	5.735E-04	1.225E-04	7.371E-04	8.089E-03
13	1.964E-04	3.944E-05	2.524E-04	3.080E-03
14	6.017E-05	1.149E-05	7.733E-05	1.026E-03
15	1.680E-05	3.073E-06	2.159E-05	3.079E-04

Table 3. EOC for the case 2.1.1

$k$	$x = 0.4$	$x = 0.6$
1	1.036659	1.043486
2	1.096530	1.053197
3	1.116673	1.187045



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

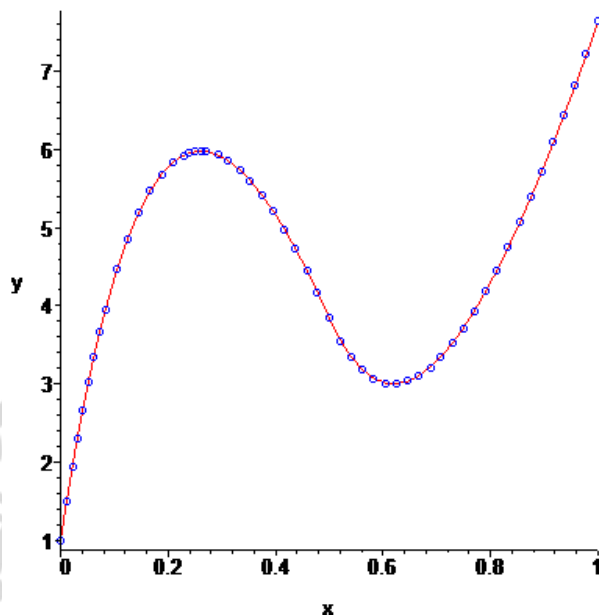


Fig. 1. Continuous line:  $u_{Exact}(x)$ , o: VIM,  $\lambda = 100$ ,  $k = 2$

2.1.2. Taking  $\lambda = 10$ ,  $k = 1$ , and

$$f(x, u) = U(x - 1) = \begin{cases} 0, & \text{if } x < 1 \\ 1, & \text{if } x \geq 1 \end{cases}$$

the unit step function at  $x = 1$ . We calculate the first iterations from the Eq. (2.4)

$$u_0(x) = 1,$$

$$u_1(x) = \begin{cases} 1 - x, & x < 1 \\ -9 + 9x, & x \geq 1 \end{cases}$$

$$u_2(x) = \begin{cases} 1 - x + \frac{1}{2}x^2, & x < 1 \\ -14 + 19x - \frac{9}{2}x^2, & x \geq 1 \end{cases}$$

$$u_3(x) = \begin{cases} 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3, & x < 1 \\ -\frac{47}{3} + 24x - \frac{19}{2}x^2 + \frac{3}{2}x^3, & x \geq 1 \end{cases}$$

$$u_4(x) = \begin{cases} 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4, & x < 1 \\ -\frac{193}{12} + \frac{77}{3}x - 12x^2 + \frac{19}{6}x^3 - \frac{3}{8}x^4, & x \geq 1 \end{cases}$$



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الأول 2020



الجامعة العراقية  
مركز البحوث والدراسات

:

thus, in this way, the rest of the iterations can be obtained. When  $n \rightarrow \infty$ , the closed form is

$$u_{Exact}(x) = \begin{cases} e^{-x}, & x < 1 \\ 10 - 10e^{1-x} + e^{-x}, & x \geq 1 \end{cases} \quad (2.6)$$

which is the analytic solution for the case 2.1.2.

In Table 4 show a comparison of the numerical results applying the VIM ( $n = 9$ ), the numerical solution of (2.4) with the Trapezoidal rule (TRAP) and the numerical solution of (2.4) with the Simpson rule (SIMP) with the analytic solution (2.6) and HAM [11]. Twenty points have been utilized in the Trapezoidal rule and the Simpson rule. It can be noticed that the results obtained by the present method are compatible with the HAM [11]. In Table 5 we list the MAE,  $\|\cdot\|_2$ , the MRE and the MRR obtained by the VIM with the analytic solution (2.6) on the interval  $[0,2]$ . The EOC are 1.0793 at  $x = 0.9$  and 1.0896 at  $x = 1.1$ .

In Fig. 2 illustrates both the analytic solution  $u_{Exact}(x)$  and our approximation by VIM ( $n = 8$ ) within the interval  $0 \leq x \leq 2$ .

In this case, the VIM is applicable until the value  $k \simeq 2$ .

“Table 4. Numerical results for the case 2.1.2”

$x$	$u_{Exact}(x)$	VIM	HAM [11]	TRAP	SIMP
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.2	0.818730753	0.818730753	0.818730753	0.818730753	0.818730753
0.4	0.670320046	0.670320046	0.670320046	0.670320046	0.670320046
0.6	0.548811636	0.548811634	0.548811662	0.548811634	0.548811634
0.8	0.449328964	0.449328936	0.449329306	0.449328938	0.449328938
1.0	0.367879441	0.367879188	0.367881944	0.367879214	0.367879213
1.2	2.113886681	2.113885143	2.113899362	2.113885334	2.113885330
1.4	3.543396503	3.543389440	3.543446369	3.543390478	3.543390455
1.6	4.713780157	4.713753757	4.713942852	4.713758264	4.713758164
1.8	5.672009247	5.671925111	5.672468038	5.671941556	5.671941191
2.0	6.456540871	6.456305114	6.457688492	6.456357320	6.456356171



جامعة دهوك  
كلية التربية الاساس



الجامعة العراقية  
مركز البحوث والدراسات

ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني  
"المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني"  
17-16 كانون الاول 2020 (المجلد الثاني)







جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

( المجلد الثاني ) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

“Table 5. MAE,  $\|\cdot\|_2$ , MRE and MRR for the case 2.1.2”

$m$	MAE	$\ \cdot\ _2$	MRE	MRR
4	1.267E-01	6.671E-02	1.963E-02	2.696E-01
5	5.654E-02	2.427E-02	8.757E-03	1.833E-01
6	1.845E-02	7.056E-03	2.858E-03	7.499E-02
7	4.953E-03	1.743E-03	7.672E-04	2.341E-02
8	1.147E-03	3.783E-04	1.777E-04	6.101E-03
9	2.357E-04	7.355E-05	3.651E-05	1.383E-03

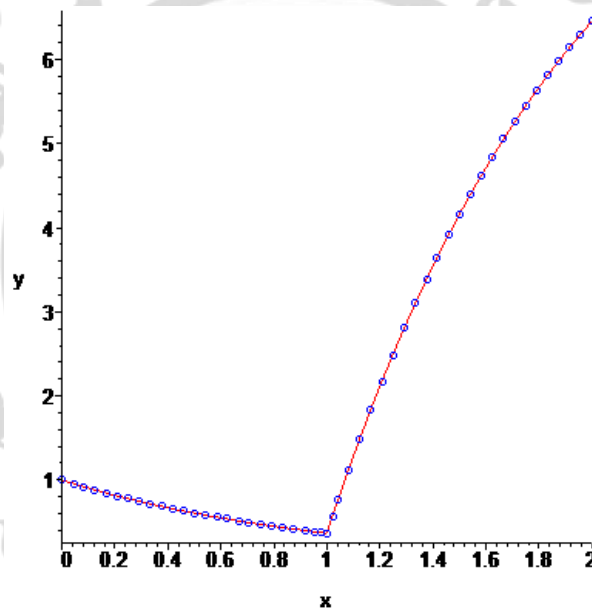


Fig. 2. Continuous line:  $u_{Exact}(x)$ , o: VIM

**2.1.3.** Letting  $k = 1$ ,  $\lambda = 1$  and  $f(x, u) = \delta(x - 1)$ , the unit impulse function at  $x = 1$ . We calculate the first iterations from the Eq. (2.4)

$$u_0(x) = 1,$$

$$u_1(x) = U(x - 1) + 1 - x,$$

$$u_2(x) = (2 - x)U(x - 1) + 1 - x + \frac{1}{2}x^2,$$

$$u_3(x) = \left(\frac{5}{2} - 2x + \frac{1}{2}x^2\right)U(x - 1) + 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3,$$



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

$$u_4(x) = \left(\frac{8}{3} - \frac{5}{2}x + x^2 - \frac{1}{6}x^3\right)U(x-1) + 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4,$$

:

thus, in this way, the rest of the iterations can be obtained. When  $n \rightarrow \infty$ , the closed form is

$$u_{Exact}(x) = U(x-1)e^{-x+1} + e^{-x}, \quad (2.7)$$

which is the analytic solution of the case 2.1.3.

In Table 6 we list the MAE,  $\|\cdot\|_2$ , the MRE and the MRR obtained by the VIM with the analytic solution (2.7) on the interval [0,2].

The EOC are 1.0793 at  $x = 0.9$  and 1.0907 at  $x = 1.1$ .

Fig. 3 illustrates both the analytic solution  $u_{Exact}(x)$  and our approximation by VIM ( $n = 8$ ) within the interval  $0 \leq x \leq 2$ .

Again, in this case, the VIM is applicable for the values  $k \lesssim 2.2$ .

“Table 6. MAE,  $\|\cdot\|_2$ , MRE and MRR for the case 2.1.3”

$m$	MAE	$\ \cdot\ _2$	MRE	MRR
4	1.634E-01	7.522E-02	3.248E-01	5.000E-01
5	6.154E-02	2.532E-02	1.223E-01	2.249E-01
6	1.900E-02	7.158E-03	3.777E-02	8.055E-02
7	5.000E-03	1.750E-03	9.936E-03	2.400E-02
8	1.150E-03	3.786E-04	2.286E-03	6.150E-03
9	2.357E-04	7.352E-05	4.685E-04	1.386E-03



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

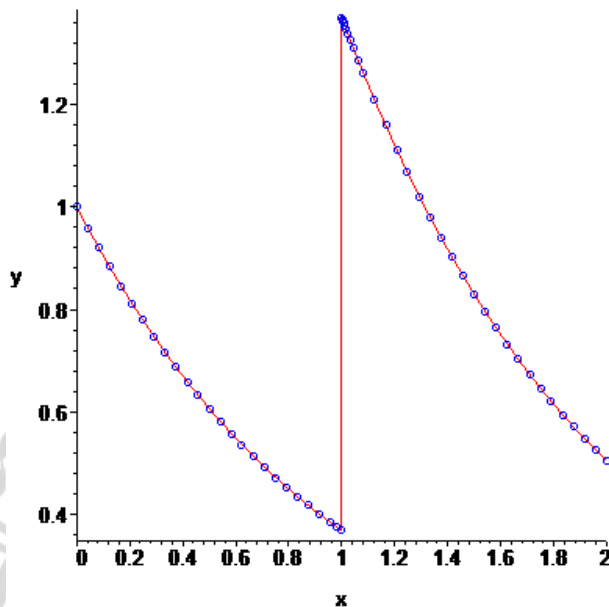


Fig. 3. Continuous line:  $u_{Exact}(x)$ , o: VIM

**3.1.4.** Now we take  $k = 1$ ,  $\lambda = 1$  and

$$f(x, u) = \delta\left(x - \frac{1}{2}\right) + \delta(x - 1) + \delta\left(x - \frac{3}{2}\right),$$

the unit impulse function at  $x = \frac{1}{2}, 1, \frac{3}{2}$ . We calculate the first iterations from the Eq. (2.4)

$$u_0(x) = 1,$$

$$u_1(x) = U\left(x - \frac{1}{2}\right) + U(x - 1) + U\left(x - \frac{3}{2}\right) + 1 - x,$$

$$u_2(x) = \left(\frac{3}{2} - x\right)U\left(x - \frac{1}{2}\right) + (2 - x)U(x - 1) + \left(\frac{5}{2} - x\right)U\left(x - \frac{3}{2}\right) + 1 - x + \frac{1}{2}x^2,$$

$$u_3(x) = \left(\frac{13}{8} - \frac{3}{2}x + \frac{1}{2}x^2\right)U\left(x - \frac{1}{2}\right) + \left(\frac{5}{2} - 2x + \frac{1}{2}x^2\right)U(x - 1)$$

$$+ \left(\frac{29}{8} - \frac{5}{2}x + \frac{1}{2}x^2\right)U\left(x - \frac{3}{2}\right) + 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3,$$

⋮

thus, in this way, the rest of the iterations can be obtained. When  $n \rightarrow \infty$ , the closed form is

$$u_{Exact}(x) = U\left(x - \frac{1}{2}\right)e^{\frac{1}{2}-x} + U(x - 1)e^{1-x} + U\left(x - \frac{3}{2}\right)e^{\frac{3}{2}-x} + e^{-x}, \quad (2.8)$$



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

which is the analytic solution for the case 2.1.4. The EOC for both sides of the discontinuity are given in **Table 7**. In Figs. 4 and 5 illustrate both the analytic solution  $u_{Exact}(x)$  and our approximation by VIM ( $n = 14$ ). The validity of the approximation VIM ( $n = 14$ ) only till the second discontinuity can be easily noticed in Fig. 5.

**Table 7. EOC for the case 2.1.4**

$x = 0.4$	$x = 0.6$	$x = 0.9$	$x = 1.1$	$x = 1.4$	$x = 1.6$
1.0352	1.0424	1.0627	1.0802	1.0846	1.0420

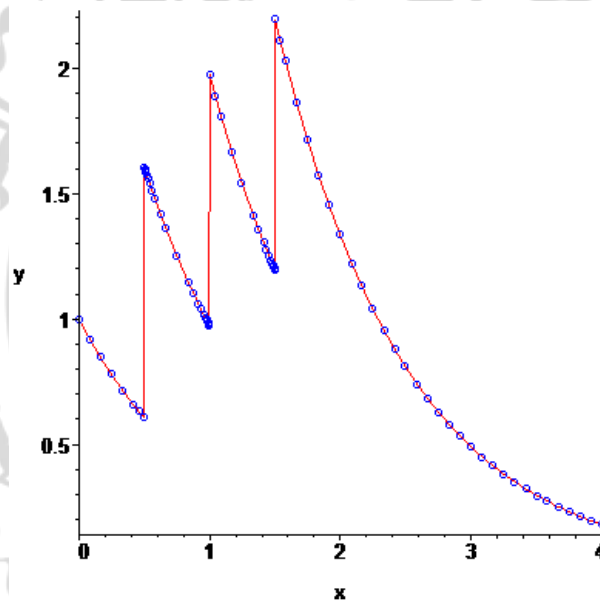


Fig. 4. Continuous line:  $u_{Exact}(x)$ , o: VIM,  $k = 1$ ,  $\lambda = 1$



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

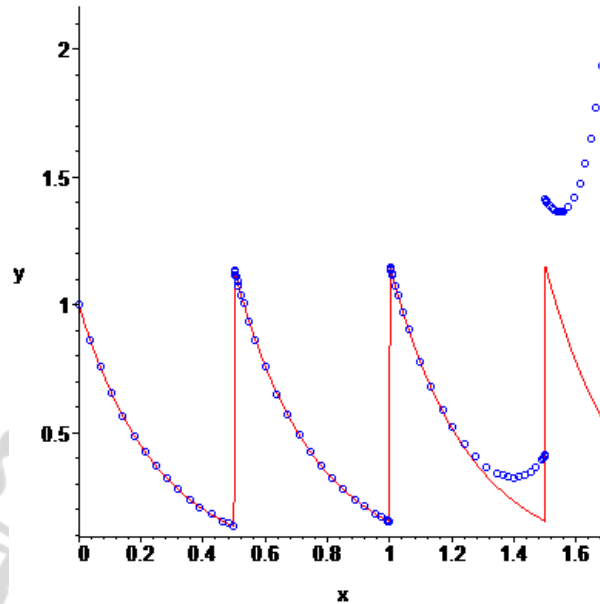


Fig. 5. Continuous line:  $u_{Exact}(x)$ , o: VIM,  $k = 2$ ,  $\lambda = 1$

**2.2 Non-linear case:** Let  $g(u) = u^2$  and  $\alpha = 1$ .

**2.2.1.** If we are taking  $k = 1$ ,  $\lambda = 1$  and

$$f(x, u) = U(x - 1) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1 \end{cases}$$

the unit step function at  $x = 1$ . We calculate the first iterations from the Eq. (2.4)

$$u_0(x) = 1,$$

$$u_1(x) = \begin{cases} 1 & x \leq 1 \\ x & x > 1 \end{cases}$$

$$u_2(x) = \begin{cases} 1, & x \leq 1 \\ \frac{1}{6} + x - \frac{1}{2}x^2 + \frac{1}{3}x^3, & x > 1 \end{cases}$$

$$u_2(x) = \begin{cases} 1, & x \leq 1 \\ \frac{239}{1260} + \frac{31}{36}x - \frac{1}{3}x^2 + \frac{4}{9}x^3 - \frac{11}{36}x^4 + \frac{11}{60}x^5 - \frac{1}{18}x^6 + \frac{1}{63}x^7, & x > 1 \end{cases}$$

⋮

thus, in this way, the rest of the iterations can be obtained.

Fig. 6 illustrates both the numeric solution  $u_N(x)$  with a very small error and our approximation by VIM ( $n = 9$ ) for  $0 \leq x \leq 2$ .





جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

For all values of  $\lambda$ , the VIM is applicable in this case, when  $k \lesssim 1.5$ .

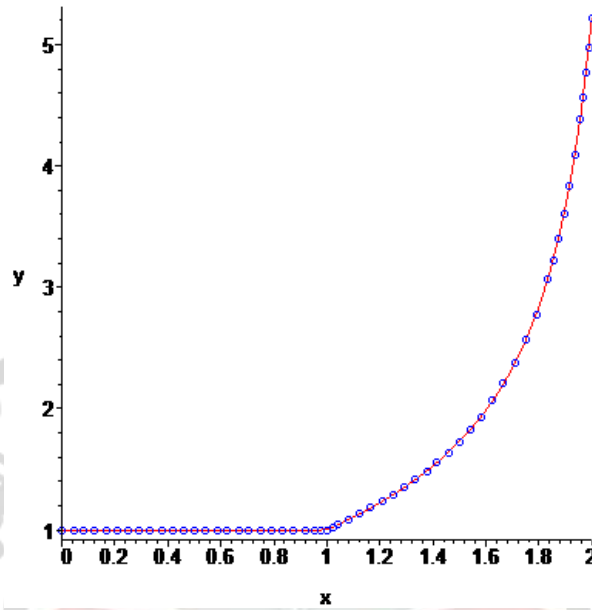


Fig. 6. Continuous line:  $u_N(x)$ , o: VIM

**2.2.2.** Taking  $k = 1$ ,  $\lambda = 1$  and  $f(x, u) = \delta\left(x - \frac{1}{2}\right)$ , the unit impulse function at  $x = \frac{1}{2}$ . We calculate the first iterations from the Eq. (2.4)

$$u_0(x) = 1,$$

$$u_1(x) = 1 + U\left(x - \frac{1}{2}\right),$$

$$u_2(x) = 1 + 2xU\left(x - \frac{1}{2}\right),$$

$$u_3(x) = 1 + \left(\frac{7}{12} + x^2 + \frac{4}{3}x^3\right)U\left(x - \frac{1}{2}\right),$$

⋮

thus, in this way, the rest of the iterations can be obtained.

In Fig. 7 we show our approximations by VIM ( $n = 8$ ) and VIM ( $n = 9$ ) for  $0 \leq x \leq 1$ .

For any values of  $\lambda$ , the VIM is applicable in this case, when  $k \lesssim 2.3$ .



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

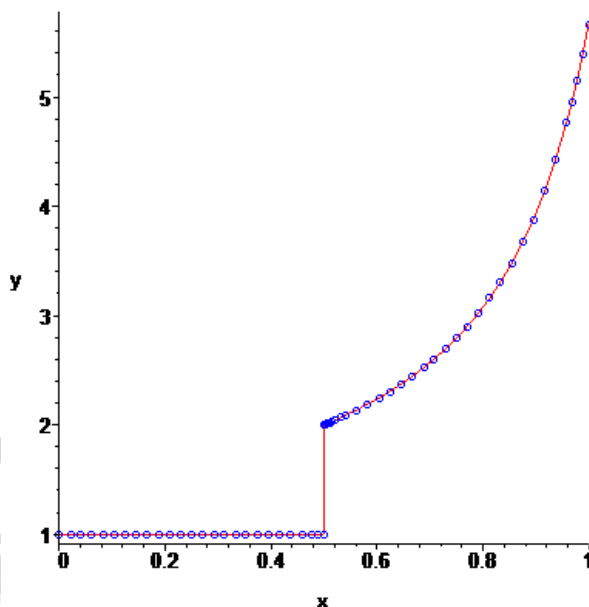


Fig. 7. Continuous line: VIM ( $n = 8$ ) o: VIM ( $n = 9$ )

2.2.3. Finally, we take  $k = 1$ ,  $\lambda = 1$  and

$$f(x, u) = \delta\left(x - \frac{1}{4}\right) + \delta\left(x - \frac{1}{2}\right),$$

the unit impulse function at  $x = \frac{1}{4}, \frac{1}{2}$ . We calculate the first iterations from the Eq. (2.4)

$$u_0(x) = 1,$$

$$u_1(x) = 1 + U\left(x - \frac{1}{4}\right) + U\left(x - \frac{1}{2}\right),$$

$$u_2(x) = 1 + \left(\frac{1}{2} + 2x\right)U\left(x - \frac{1}{4}\right) + (-1 + 4x)U\left(x - \frac{1}{2}\right)$$

$$u_3(x) = 1 + \left(\frac{2}{3} + \frac{3}{4}x + 2x^2 + \frac{4}{3}x^3\right)U\left(x - \frac{1}{4}\right) + \left(\frac{2}{3} - x - 2x^2 + \frac{32}{3}x^3\right)U\left(x - \frac{1}{2}\right)$$

⋮

thus, in this way, the rest of the iterations can be obtained.

Fig. 8 represents our approximations by VIM ( $n = 8$ ) and VIM ( $n = 9$ ) for  $0 \leq x \leq 0.6$ .



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الاول 2020



الجامعة العراقية  
مركز البحوث والدراسات

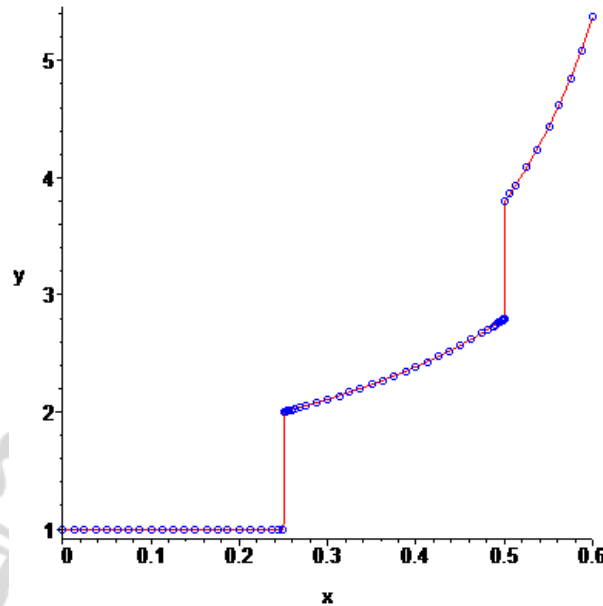


Fig. 8. Continuous line: VIM ( $n = 8$ ) o: VIM ( $n = 9$ )

## Conclusions

In this paper, the VIM has been successfully applied to solve IVPs of first order with discontinuities. The convergence of the method does not affect by the size of the jump (given by  $\lambda$ ) on both sides of the discontinuity. With  $k = 3$ , the application by the VIM does not converge even with small values of the parameter such as  $\lambda = 10^{-3}$ .

The proposed scheme for VIM was used straight without the need for transformation equations or restricted assumptions. The VIM approach was tested by using the method to obtain approximate analytic solutions of the linear case. The results obtained by the current method VIM are convenient with ADM and HAM. The results obtained in all cases explain the accuracy and efficacy of this method.

## Acknowledgements

This work was supported by the College of Computer Science and Mathematics, University of Mosul, Mosul, Iraq.

## References



جامعة دهوك  
كلية التربية الاساس



ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني

”المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني“

(المجلد الثاني) 17-16 كانون الأول 2020



الجامعة العراقية  
مركز البحوث والدراسات

- [1] M. Inokuti, H. Sekine, T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics, in: Variational Method in the Mechanics of solids, Pergamon Press, New York, 1978, pp. 156–162.
- [2] Ji-Huan He, Variational iteration method—a kind of non-linear analytical technique: some examples, *International Journal of Non-Linear Mechanics* **34** (1999), 699-708.
- [3] Ji-Huan He, Variational iteration method for autonomous ordinary differential systems, *Applied Mathematics and Computation* **114** (2000), 115-123.
- [4] Ji-Huan He, Variational iteration method—Some recent results and new interpretations, *Journal of Computational and Applied Mathematics* **207** (2007), 3–17.
- [5] Abdul-Majid Wazwaz , A comparison between the variational iteration method and Adomian decomposition Method, *Journal of Computational and Applied Mathematics* **207** (2007), 129–136.
- [6] Abdul-Majid Wazwaz, The variational iteration method for analytic treatment for linear and nonlinear ODEs, *Applied Mathematics and Computation* **212** (2009), 120–134. doi:10.1016/j.amc.2009.02.003
- [7] L. Casasús and W. Al-Hayani, The decomposition method for ordinary differential equations with discontinuities, *Applied Mathematics and Computation* **131** (2002), 245-251.
- [8] W. Al-Hayani and L. Casasús, On the applicability of the Adomian method to initial value problems with discontinuities, *Applied Mathematics Letters* **19** (2006), 22-31.
- [9] W. Al-Hayani and L. Casasús, The Adomian Decomposition Method for Boundary Value Problems with Discontinuities, *Mathematical Sciences Letters* **6** (1) (2017), 1-7.
- [10] Waleed Al-Hayani, Rasha Fahad, Homotopy Analysis Method for Solving Initial Value Problems of Second Order with Discontinuities, *Applied Mathematics* **10** (2019), 419-434. <https://doi.org/10.4236/am.2019.106030>
- [11] Waleed Al-Hayani; Rasha Fahad, Applying the homotopy analysis method to initial value problems with discontinuities, *Journal of Physics: Conference Series*, The Sixth International Scientific Conference of Al-Khwarizmi Society, 22-23 November 2020.



جامعة دهوك  
كلية التربية الاساس



الجامعة العراقية  
مركز البحوث والدراسات

ابحاث المؤتمر العلمي الدولي الرابع المشترك الثاني  
"المستجدات الحديثة في التعليم العالي في ظل التعليم الالكتروني"  
17-16 كانون الاول 2020 (المجلد الثاني)

- [12] Ji-Huan He, The Homotopy perturbation method for nonlinear oscillators with discontinuities, *Applied Mathematics and Computation* 151 (1) (2004), 287-292.
- [13] B. A. Finlayson, The method of Weighted residuals and variational principles, Academic Press, New York, 1972.

